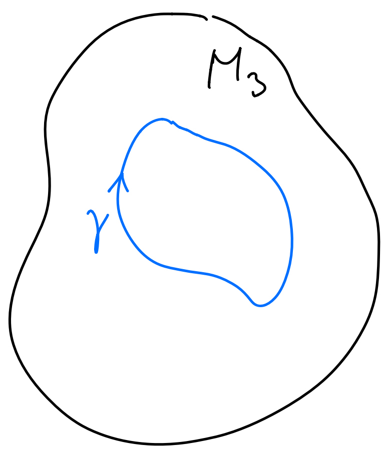
Wilson Lines:

An interesting class of observables is the one of "line defects/operators", also called "Wilson lines":

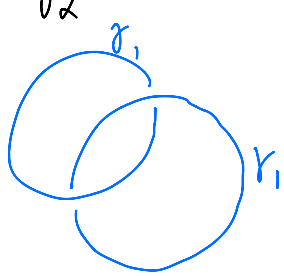


oriented loop $\gamma \hookrightarrow M_3$, rep. R of G

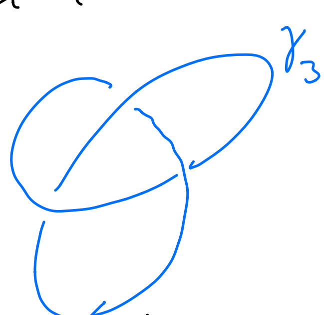
$$\to \quad W(R,\gamma) := \mathrm{Tr}_R\left(P\exp\oint_\gamma A\right)$$

$$\left\langle \prod_\alpha W(R_\alpha,\gamma_\alpha)\right\rangle = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A \prod_\alpha W(R_\alpha,\gamma_\alpha)\, e^{2\pi i CS(A)}$$

The γ_α could form a link or knot:



link



Knot

$G = SU(2)$

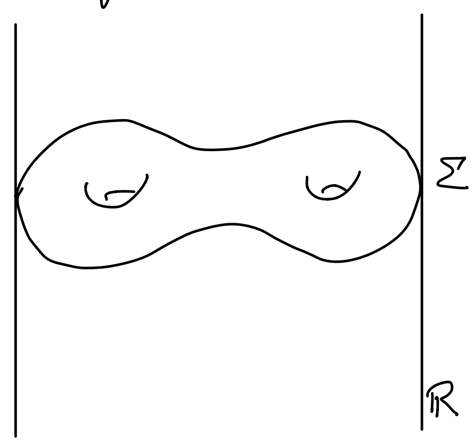
$\to \langle W(\underline{2}, \gamma)\rangle_{M_3 = S^3}$ = polynomial in $q = e^{\frac{2\pi i}{k+2}}$

"Jones polynomial"

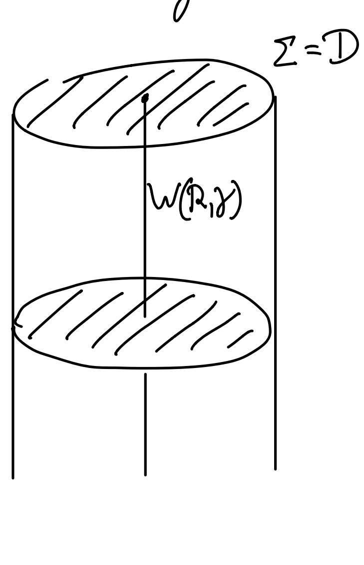
Recall "Holographic" relation to 2D (rational) conformal field theory:

$M_3 = \Sigma \times \mathbb{R}$

$\mathcal{H}(\Sigma)$ finite dim. space of "conformal blocks" of 2D CFT (WZW)



Introducing Wilson lines to this picture, we get



On Σ we decompose $d = dt\frac{\partial}{\partial t} + \tilde{d}$ and $A = A_0 + \tilde{A}$

$\longrightarrow$ constraint: $\tilde{F} = \tilde{d}\tilde{A} + \tilde{A}^2 = 0$ (without Wilson lines)

$\to$ solved by: $\tilde{A} = -\tilde{d}UU^{-1}$, $U: D\times\mathbb{R} \to G$

In terms of the U's the Chern-Simons action becomes (ϕ is the angular coordinate on ∂D):

$$2\pi\, CS(A) \to \frac{K}{2\pi} \int_{\partial M_3} \mathrm{Tr}\left(U^{-1}\partial_\phi U\, U^{-1}\partial_t U\right) d\phi\, dt$$

$$+ \frac{K}{12\pi} \int_{M_3} \mathrm{Tr}\left(U^{-1} dU\right)^3 \qquad (*)$$

The above action is invariant under transformations on the boundary:

$$U(\phi, t) \longmapsto \tilde{V}(\phi)\, U\, V(t) \qquad \text{(exercise)}$$

$(*) \longrightarrow$ recover chiral version of WZW model

(invariance under $\tilde{V}$ is global sym.)

inclusion of Wilson loops amounts to adding the following term to the action:

$$\int dt\, \mathrm{Tr}\, \lambda\, w^{-1}(\partial_0 + A_0)\, w(t)$$

where $\lambda = \vec{\lambda}\cdot\vec{H}$ is a weight and the action has gauge invariance $w(t) \mapsto w(t)\, h(t)$

→ integrating out $w(t)$ gives back the Wilson loop $\mathrm{Tr}_{R_\lambda} P\exp\left(\int_t A_0 dt\right)$

(arxiv/1401.6167)

→ the constraint now becomes:

$$\frac{k}{2\pi}\tilde{F}(x) + w(t)\lambda w^{-1}(t)\, \delta^{(2)}(x-P) = 0$$

position of Wilson line on D.

→ solved by

$$\tilde{A} = -\tilde{\partial}\, \tilde{U}\, \tilde{U}^{-1}$$

where $\tilde{U} = U \exp\left(\frac{1}{k} w(t)\lambda w^{-1}(t)\phi\right)$

where U commutes with $w(t)\lambda w^{-1}(t)$

→ conjugacy class of holonomy of flat connection around P is determined by the representation at P.

inserting into the CS-action gives:

$$CS(A) \to k\, S_{CWZW}(U) + \frac{1}{2\pi}\int_{\partial M} \mathrm{Tr}\, \lambda U^{-1}\partial_0 U \quad (**)$$

→ invariant under
$$U(\phi, t) \mapsto \bar{V}(\phi)\, U V(t)$$

where $V(t)$ commutes with λ

Quantization of (**) gives integrable highest weight module $\mathcal{H}_\lambda$.

§9. Conformal field theory and the Jones polynomial

A "link" L is an embedding

$$f : S^1 \cup \cdots \cup S^1 \longrightarrow S^3$$

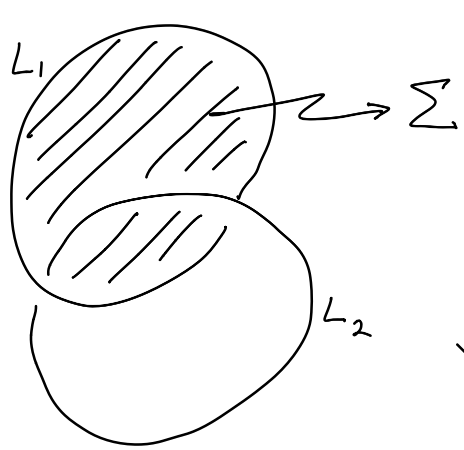
The image of each S^1 is called "link component"

$\longrightarrow L = L_1 \cup L_2 \cdots \cup L_m$

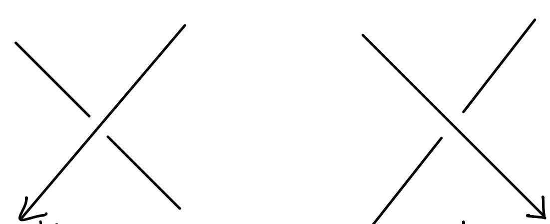
A link L with one component is a "knot".

Let $L = L_1 \cup L_2$ be a link with two components

The "likinging number" $lk(L_1, L_2)$ is the intersection number of an oriented surface Σ_1 in S^3 s.t. $\partial \Sigma_1 = L_1$ with L_2



$$lk(L_1, L_2) = \frac{1}{2}(\#\text{positive crossings} - \#\text{negative crossings})$$



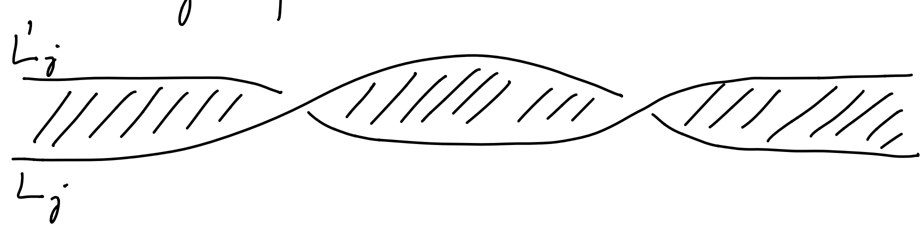
positive crossing

negative crossing

A "framing" of a link L is an integer n_j for each component L_j given by

$$n_j = lk(L_j, L_j')$$

where L_j' is a simple closed curve on the boundary of a tubular neighborhood of L_j

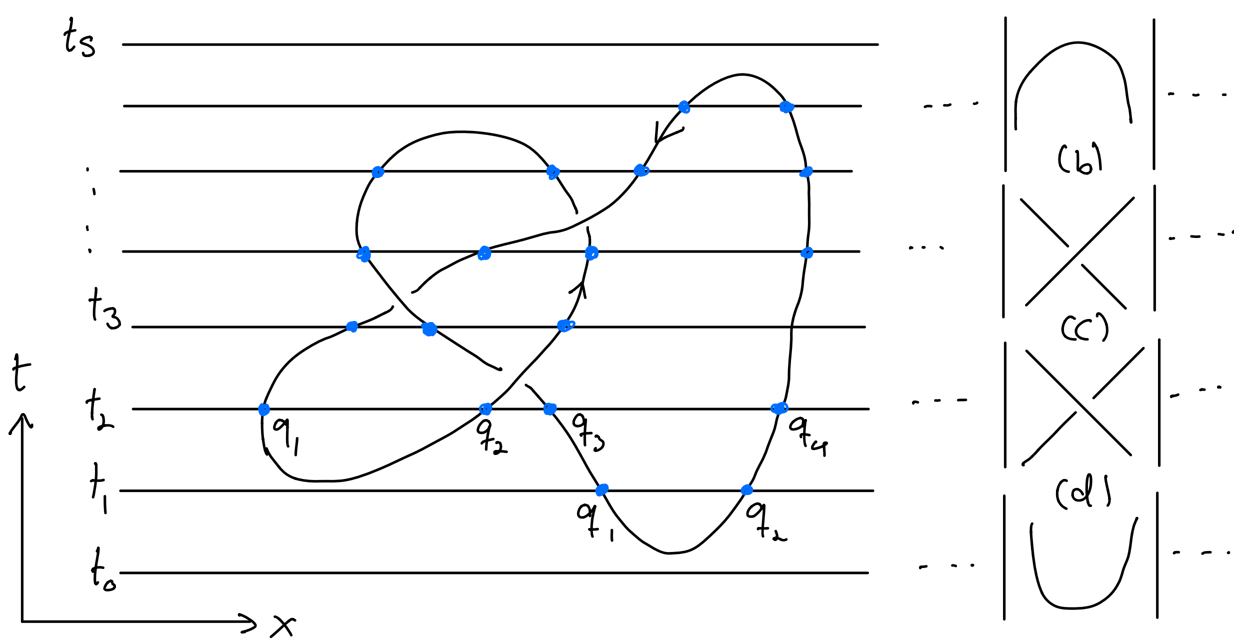


$\longrightarrow n_j = 1$

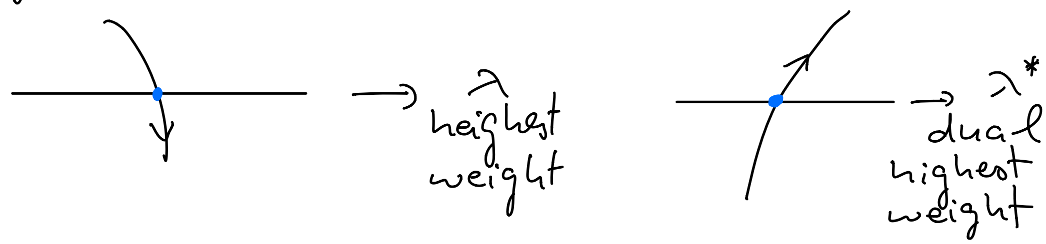
Let L be an oriented framed link in $\mathbb{R}^3$.

$\longrightarrow$ associate level K highest weights $\lambda_1, \ldots, \lambda_m$ to $L_1, \ldots, L_m$

We split each L into "elementary tangles":



To each q_i we associate a level K highest weight:



$\xrightarrow{\text{at } t_j}$ consider space of conformal blocks for Riemann sphere with points $q_1, \dots, q_n$ and highest weights as defined above

$\longrightarrow V(t_j)$

in particular: $V(t_0) = V(t_s) = \mathbb{C}$

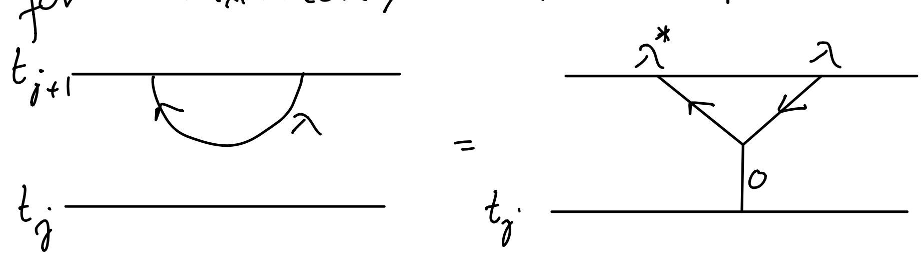
Associate a linear map:

$$Z_j : V(t_j) \longrightarrow V(t_{j+1}), \quad 0 \leq j \leq s-1$$

to each elementary tangle as follows:

(1) σ_i , σ_i^{-1} $\longrightarrow$ holonomy of KZ equation

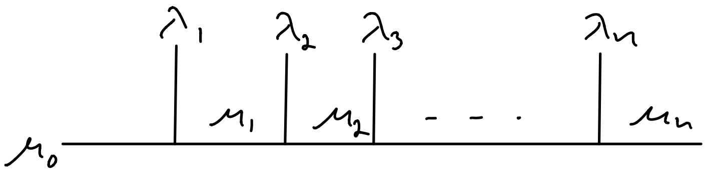
(2) for maximal/minimal points define



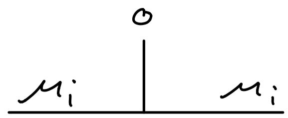
and set $V(t_j) = V_{\lambda_1 \cdots \lambda_i \lambda_{i+1} \cdots \lambda_n}$

$$V(t_{j+1}) = V_{\lambda_1 \cdots \lambda_i \lambda \lambda^* \lambda_{i+1} \cdots \lambda_n}$$

Recall that $V_{\lambda_1 \cdots \lambda_n}$ has basis



$v_{\mu_0 \cdots \mu_n}$

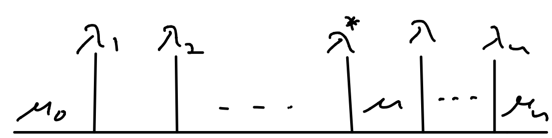
inserting 

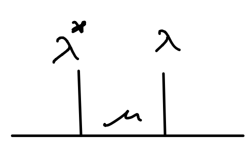
gives natural identification $V_{\lambda_1 \cdots \lambda_i \lambda_{i+1} \cdots \lambda_n} \cong V_{\lambda_1 \cdots \lambda_i 0 \lambda_{i+1} \cdots \lambda_n}$

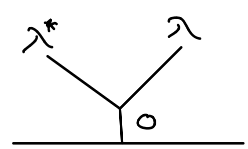
defined by $v_{\mu_0 \cdots \mu_i \cdots \mu_n} \longmapsto v_{\mu_0 \cdots \mu_i \mu_i \cdots \mu_n}$

Now define $Z_j : V(t_j) \longrightarrow V(t_{j+1})$ by

$$v_{\mu_0 \cdots \mu_i \mu_i \cdots \mu_n} \longmapsto \sum_{\mu} F_{\mu 0}\, v_{\mu_0 \cdots \mu_i \mu \mu_i \cdots \mu_n}$$

where $v_{\mu_0 \cdots \mu_i \mu \mu_i \cdots \mu_n} =$ 

and $\sum_{\mu} F_{\mu 0}$ 

$=$ 

Composing the above linear maps Z_j, $0 \le j \le s-1$

we get $Z(L; \lambda_1, \ldots, \lambda_m) = Z_{s-1} \circ \cdots \circ Z_1 \circ Z_0(1)$